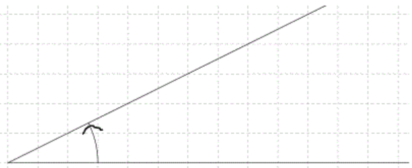
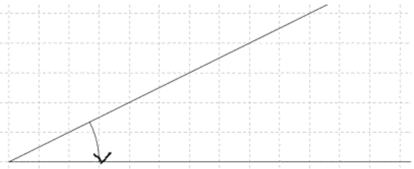
Mesura d'angles

Considerarem dues orientacions d'angles



Orientació positiva



Orientació negativa

A més, els angles poden tenir més d'una volta

$750° = 30° + 2 \cdot 360°$ $\qquad -370° = -10° - 1 \cdot 360°$

Per contar les voltes dividim l'angle per $360°$

Exemple

$$\begin{array}{r|l} 1254° & 360° \\ \hline 174 & 3 \end{array} \qquad \begin{array}{r|l} 2450° & 360° \\ \hline 290 & 6 \end{array}$$

$1254° = 174° + 3 \cdot 360°$ $\qquad -2450° = -290° - 6 \cdot 360°$

Definim

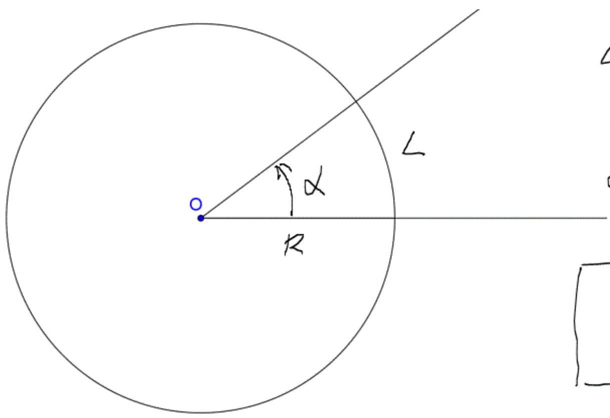
$$\boxed{\begin{aligned} 1 \text{ volta} &= 360° \\ 1° &= 60' \\ 1' &= 60'' \end{aligned}}$$

Exemples

$2{,}45° = 2° + 0{,}45° = 2° + 0{,}45 \cdot 60' = 2° + 27' = 2°27'$

$3°40'10'' = 3° + 40/60° + 10/3600° = 3{,}67°$

Mesura d'angles en radians



L: longitud de l'arc

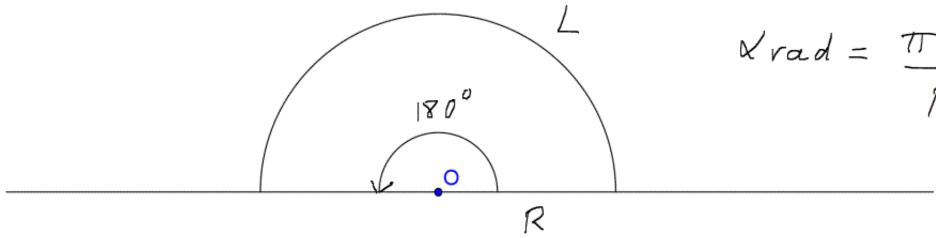
R: radi

α: mesura en radians

$$\boxed{\alpha = \frac{L}{R}}$$

Exemples

1)



$$L = \frac{2\pi R}{2} = \pi R$$

$$\alpha \text{ rad} = \frac{\pi R}{R} = \pi$$

Per tant $\boxed{180^\circ = \pi \text{ rad}}$

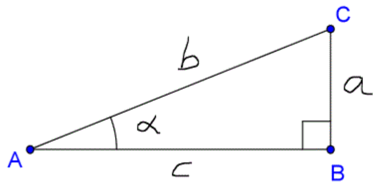
2) Tranformeu en radians 72°

$$72^\circ = 72^\circ \cdot \frac{\pi \text{ rad}}{180^\circ} = \frac{72}{180}\pi \text{ rad} = \frac{2\pi}{5} \text{ rad}$$

3) Transformeu en graus $\frac{5\pi}{6}$ rad

$$\frac{5\pi}{6} \text{ rad} = \frac{5\pi}{6} \text{ rad} \cdot \frac{180^\circ}{\pi \text{ rad}} = \frac{5}{6} \cdot 180^\circ = 150^\circ$$

Raons trigonomètriques d'un angle agut



$$\sin\alpha = \frac{\text{catet oposat}}{\text{hipotenusa}} = \frac{a}{b}$$

$$\cos\alpha = \frac{\text{catet contigu}}{\text{hipotenusa}} = \frac{c}{b}$$

$$\tan\alpha = \frac{\text{catet oposat}}{\text{catet contigu}} = \frac{a}{c}$$

Propietats

1) $\sin^2\alpha + \cos^2\alpha = 1$

2) $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$

3) $1 + \tan^2\alpha = \dfrac{1}{\cos^2\alpha}$

Demostració

1) T. Pitàgores $b^2 = a^2 + c^2$

$$1 = \frac{a^2}{b^2} + \frac{c^2}{b^2}$$

$$1 = \left(\frac{a}{b}\right)^2 + \left(\frac{c}{b}\right)^2$$

Per tant $1 = \sin^2\alpha + \cos^2\alpha$

2) $\tan\alpha = \dfrac{a}{c} = \dfrac{a/b}{c/b} = \dfrac{\sin\alpha}{\cos\alpha}$

3) $1 = \sin^2\alpha + \cos^2\alpha$

$$\frac{1}{\cos^2\alpha} = \frac{\sin^2\alpha}{\cos^2\alpha} + 1$$

$$\frac{1}{\cos^2\alpha} = \left(\frac{\sin\alpha}{\cos\alpha}\right)^2 + 1$$

$$\frac{1}{\cos^2\alpha} = \tan^2\alpha + 1$$

Altres raons trigonomètriques

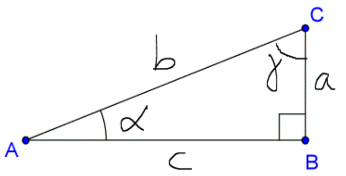
$\sec\alpha = \dfrac{1}{\cos\alpha}$ secant de l'angle α

$\operatorname{cosec}\alpha = \dfrac{1}{\sin\alpha}$ cosecant de l'angle α

$\operatorname{cotan}\alpha = \dfrac{1}{\tan\alpha}$ cotangent de l'angle α

Raons trigonomètriques de l'angle complementari

$\gamma = 90° - \alpha$



$\sin\gamma = \dfrac{c}{b} = \cos\alpha$

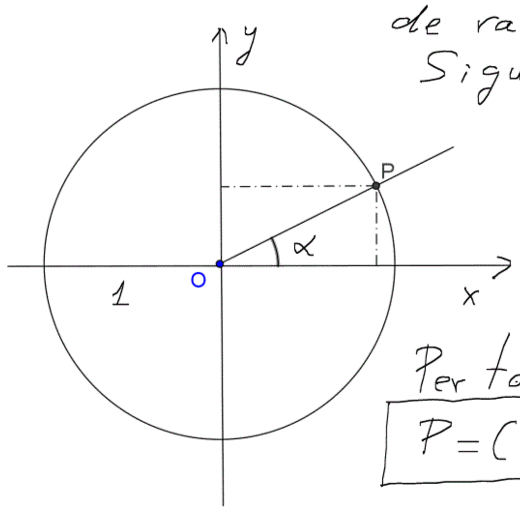
$\cos\gamma = \dfrac{a}{b} = \sin\alpha$

$\tan\gamma = \dfrac{c}{a} = \dfrac{1}{a/c} = \operatorname{cotan}\alpha$

$\sin(90° - \alpha) = \cos\alpha$
$\cos(90° - \alpha) = \sin\alpha$
$\tan(90° - \alpha) = \operatorname{cotan}\alpha$

Raons trigonomètriques d'un angle qualsevol

Considero la circumferència de radi 1



Sigui $P = (x, y)$ aleshores

$\cos\alpha = \dfrac{x}{1} = x$

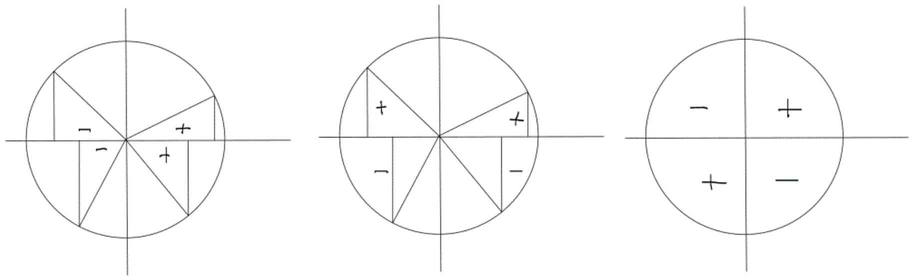
$\sin\alpha = \dfrac{y}{1} = y$

Per tant

$$\boxed{P = (\cos\alpha, \sin\alpha)}$$

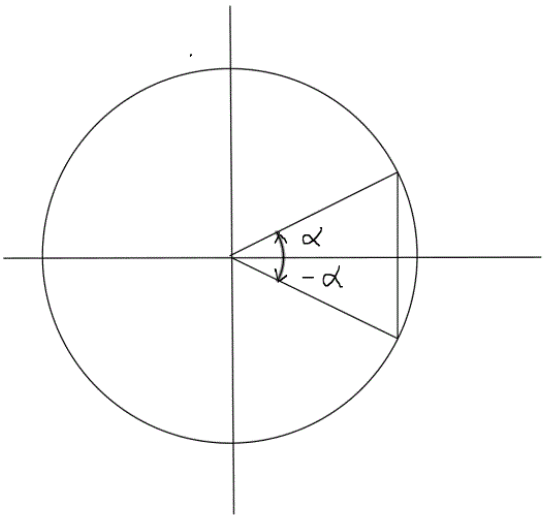
$\cos\alpha$ és l'abscissa del punt P
$\sin\alpha$ és l'ordenada del punt P

Signe del sinus, el cosinus i la tangent



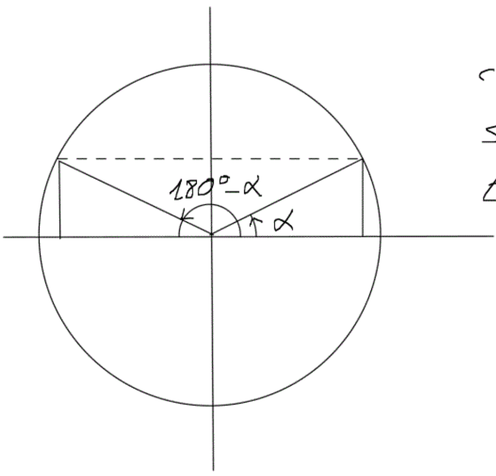
$\cos\alpha$ $\quad$ $\sin\alpha$ $\quad$ $\tan\alpha$

Raons trigonomètriques de l'angle oposat



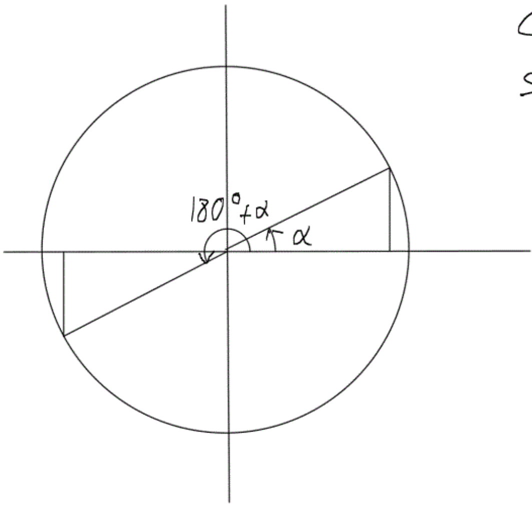
$$\cos(-\alpha) = \cos\alpha$$
$$\sin(-\alpha) = -\sin\alpha$$
$$\tan(-\alpha) = -\tan\alpha$$

Raons trigonomètriques de l'angle suplementari



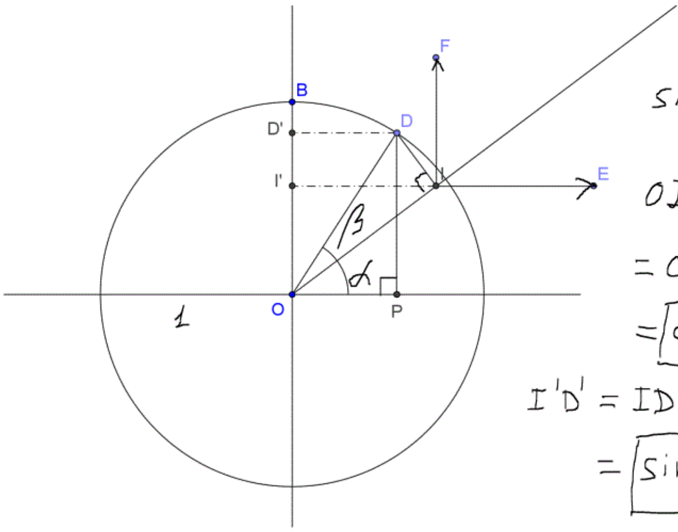
$$\cos(180^\circ-\alpha) = -\cos\alpha$$
$$\sin(180^\circ-\alpha) = \sin\alpha$$
$$\tan(180^\circ-\alpha) = -\tan\alpha$$

Raons trigonomètriques de $180° + \alpha$



$$\cos(180° + \alpha) = -\cos\alpha$$
$$\sin(180° + \alpha) = -\sin\alpha$$
$$\tan(180° + \alpha) = \tan\alpha$$

Sinus de la suma d'angles



$$\sin(\alpha+\beta) = PD = OI' + I'D'$$

$$OI' = OI\cos\widehat{BOI} =$$
$$= OI\cos(90° - \alpha) =$$
$$= \boxed{\cos\beta \cdot \sin\alpha}$$

$$I'D' = ID\cos\widehat{FID} = ID\cos\alpha =$$
$$= \boxed{\sin\beta \cdot \cos\alpha}$$

Per tant $\boxed{\sin(\alpha+\beta) = \sin\alpha . \cos\beta + \cos\alpha . \sin\beta}$

$$\sin(\alpha-\beta) = \sin(\alpha+(-\beta)) = \sin\alpha \cos(-\beta) + \cos\alpha . \sin(-\beta) =$$
$$= \sin\alpha . \cos\beta - \cos\alpha \sin\beta$$
$$\cos(\alpha+\beta) = \sin(90^\circ-(\alpha+\beta)) = \sin(90^\circ-\alpha-\beta) =$$
$$= \sin(90^\circ-\alpha)\cos\beta - \cos(90^\circ-\alpha)\sin\beta = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos(\alpha+(-\beta)) = \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta) =$$
$$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} =$$
$$= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha . \tan\beta}$$
$$\tan(\alpha-\beta) = \tan(\alpha+(-\beta)) = \frac{\tan\alpha + \tan(-\beta)}{1-\tan\alpha\tan(-\beta)} =$$
$$= \frac{\tan\alpha - \tan\beta}{1+\tan\alpha . \tan\beta}$$

<u>Exemple</u>

Calculeu les raons de 75°

$$\cos(75^\circ) = \cos(30^\circ+45^\circ) = \cos 30^\circ \cos 45^\circ - \sin 30^\circ \sin 45^\circ =$$
$$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$\sin 75^\circ = \sin(30^\circ+45^\circ) = \cos 30^\circ \sin 45^\circ + \sin 30^\circ \cos 45^\circ =$$
$$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

$$\tan 75^\circ = \tan(30^\circ+45) = \frac{\tan 30^\circ + \tan 45^\circ}{1-\tan 30^\circ \tan 45^\circ} = \frac{\frac{\sqrt{3}}{3}+1}{1-\frac{\sqrt{3}}{3}.1} =$$
$$= \frac{\sqrt{3}+3}{3-\sqrt{3}} = \frac{(\sqrt{3}+3)^2}{3^2-3} = \frac{12+6\sqrt{3}}{6} = 2+\sqrt{3}$$

Raons trigonomètriques de l'angle doble

$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha$

$\sin(2\alpha) = 2\sin\alpha . \cos\alpha$

$$\tan(2\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

Demostració

$\cos(2\alpha) = \cos(\alpha+\alpha) = \cos\alpha\cos\alpha - \sin\alpha\sin\alpha$

$\sin(2\alpha) = \sin(\alpha+\alpha) = \sin\alpha\cos\alpha + \cos\alpha\sin\alpha$

$$\tan(2\alpha) = \frac{\tan\alpha + \tan\alpha}{1-\tan\alpha . \tan\alpha}$$

Raons del l'angle meitat

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

Demostració

$$1 = \cos^2\left(\frac{\alpha}{2}\right) + \sin^2\left(\frac{\alpha}{2}\right)$$

$$\cos\alpha = \cos^2\left(\frac{\alpha}{2}\right) - \sin^2\left(\frac{\alpha}{2}\right)$$

Sumem les equacions

$$1+\cos\alpha = 2\cos^2\left(\frac{\alpha}{2}\right)$$

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

Restem les equacions

$$1-\cos\alpha = 2\sin^2\left(\frac{\alpha}{2}\right)$$

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$\tan\left(\frac{\alpha}{2}\right) = \frac{\sin(\alpha/2)}{\cos(\alpha/2)} = \pm\sqrt{\frac{\frac{1-\cos\alpha}{2}}{\frac{1+\cos\alpha}{2}}} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

<u>Exemple</u>

Calculeu les raons de $\frac{\pi}{8}$

$$\cos\left(\frac{\pi}{8}\right) = \cos\left(\frac{\pi/4}{2}\right) = \sqrt{\frac{1+\cos\left(\frac{\pi}{4}\right)}{2}} = \sqrt{\frac{1+\sqrt{2}/2}{2}} = \sqrt{\frac{1}{2}+\frac{\sqrt{2}}{4}}$$

$$\sin\left(\frac{\pi}{8}\right) = \sin\left(\frac{\pi/4}{2}\right) = \sqrt{\frac{1-\cos\left(\pi/4\right)}{2}} = \sqrt{\frac{1-\sqrt{2}/2}{2}} = \sqrt{\frac{1}{2}-\frac{\sqrt{2}}{4}}$$

$$\tan\left(\frac{\pi}{8}\right) = \tan\left(\frac{\pi/4}{2}\right) = \sqrt{\frac{1-\cos(\pi/4)}{1+\cos\left(\frac{\pi}{4}\right)}} = \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}}} = \sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}$$

$$\frac{\pi}{4}\text{rad} = \frac{\pi}{4}\text{rad}\ \frac{180^\circ}{\pi\,\text{rad}} = \boxed{45^\circ}$$

Transformació de sumes en productes

$$\cos(x-y) = \cos x\cos y + \sin x\sin y$$

$$\cos(x+y) = \cos x\cos y - \sin x\sin y$$

Sumem

$$\cos(x-y) + \cos(x+y) = 2\cos x\cos y$$

Restem

$$\cos(x-y) - \cos(x+y) = 2\sin x\sin y$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

Sumem

$$\sin(x+y) + \sin(x-y) = 2 \sin x \cos y$$

Restem

$$\sin(x+y) - \sin(x-y) = 2 \cos x \sin y$$

Canvi de variable

$$\begin{cases} x+y = a \\ x-y = b \end{cases} \quad \begin{cases} x = \frac{a+b}{2} \\ y = \frac{a-b}{2} \end{cases}$$

Les fòrmules anteriors es transformen

$$\cos a + \cos b = 2 \cos \frac{a+b}{2} \cos \frac{a-b}{2}$$

$$\cos a - \cos b = -2 \sin \frac{a+b}{2} \sin \frac{a-b}{2}$$

$$\sin a + \sin b = 2 \sin \frac{a+b}{2} \cos \frac{a-b}{2}$$

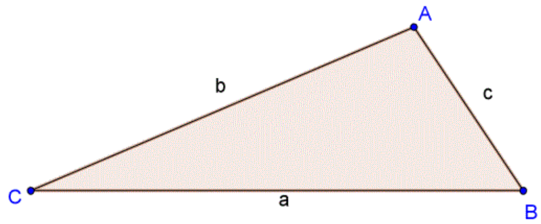
$$\sin a - \sin b = 2 \cos \frac{a+b}{2} \sin \frac{a-b}{2}$$

Exemple

Calculeu

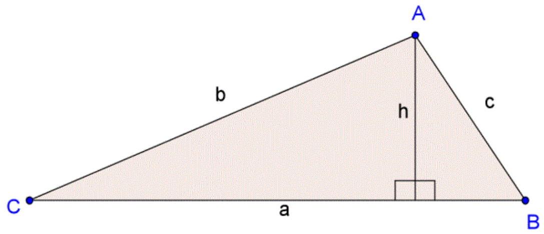
$$\cos 75^\circ + \cos 15^\circ = 2 \cos \frac{75^\circ + 15^\circ}{2} \cos \frac{75^\circ - 15^\circ}{2} = 2 \cos 45^\circ \cos 30^\circ = 2 \cdot \frac{\sqrt{2}}{2} \frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{2}$$

Teorema del Sinus



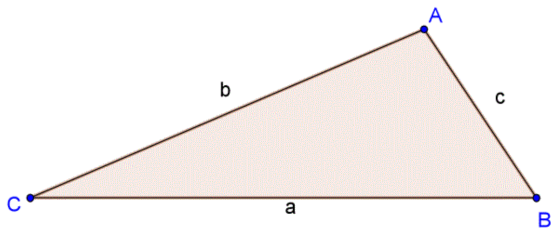
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Demostració



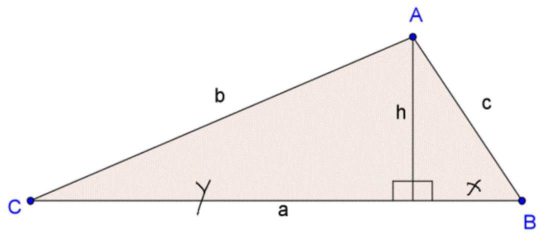
$$\left.\begin{array}{l} \sin C = \frac{h}{b} \\ \sin B = \frac{h}{c} \end{array}\right\} \Rightarrow \begin{array}{l} h = b \sin C \\ h = c \sin B \end{array} \Rightarrow b \sin C = c \sin B \Rightarrow \frac{b}{\sin B} = \frac{c}{\sin C}$$

Teorema del Cosinus



$$c^2 = a^2 + b^2 - 2ab . \cos C$$
$$a^2 = b^2 + c^2 - 2bc . \cos A$$
$$b^2 = a^2 + c^2 - 2ac . \cos B$$

<u>Demostració</u>



T. Pitàgores

$c^2 = h^2 + x^2$

$x = a - y = a - b\cos C$

$h = b.\sin C$

Substituim

$$c^2 = b^2\sin^2 C + (a - b\cos C)^2 = b^2\sin^2 C + a^2 - 2ab\cos C + b^2\cos^2 C =$$

$$= a^2 + b^2(\sin^2 C + \cos^2 C) - 2ab\cos C = a^2 + b^2 - 2ab\cos C$$